A–2 What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

Taking $10^{100} = k$, not that

$$\frac{k^{200} - 3^{200}}{k+3} = k^{199} - k^{198}\cdot 3 + k^{197}3^2 - \dots - 3^{199}.$$

Thus, it is an integer. Also note that $\frac{3^{200}}{10^{100}+3} < \frac{9^{100}}{10^{100}+3} < 1$, so $\left\lfloor \frac{10^{20k}}{10^{100}+3} \right\rfloor = \frac{10^{20k} - 3^{200}}{10^{100}+3}$. Examining the expansion, we see that -3^{199} is the only thing affecting the units digit.

$3^{199} \pmod{10} = 3^3 (81)^{49} \bmod (10)$ so it's 7. since ~~we~~ are subtracting, it will get us $\boxed{3.}$

700. Three infinite arithmetic progressions are given, whose terms are positive integers. Assuming that each of the numbers 1, 2, 3, 4, ~~5~~, 6, ~~7~~, 8 occurs in at least one of these progressions, show that 1980 necessarily occurs in one of them.

Let us use pigeon hole. If any 5 occur in one AM, clearly the difference is $\underline{1}$, so 1980 will occur (same for 6 - 8). If 4 occur in a single AM, if it's any consecutive, we are done by the above logic. If it's not consecutive we have all odds, or all evens. If it's all evens, we are also done, and if it's all odds, then the remaining AM's contain evens, so we are done.
By pigeon hole, there must be an AM w/ at least 3 numbers. We have already covered the case of the distribution 3-5-0 and 3-4-1, so 3-3-2 is left.
There are 4 even numbers,

so there must be an AM w/ atleast
2 even numbers. If there are
3, there are 2 consecutive even
numbers, so we are done. However,
having 2 even and an odd means
that it is

2e 1o

2e 1o

2o

However, if you choose, 2,6
the other becomes, 4,8,
so 1980 will get taken
on anyway.

19. (Putnam 1973) Given 9 points in 3-dimensional space with integer coordinates, show that one can select two of these points so that the segment in between them contains another point of integer coordinates.

$(a_1, b_1, c_1) \dots (a_9, b_9, c_9)$

There are 2^3 dissferent parities, so there exists 2 points such that the point's parity in each dimension is the exact same. Let these be (a_1, b_1, c_1) and (a_2, b_2, c_2)

$\left(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}, \frac{c_1+c_2}{2}\right)$ satissfies the conditions, and so we are done.